Dynamics of Compact Quantum Metric Spaces

Quantum Groups Seminar
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Based on joint work with Konrad Aguilar, Jens Kaad and Thomas Gotfredsen

The quantum way

- The Gelfand correspondence between compact Hausdorff spaces and commutative, unital C^*-algebras has given rise to a large number of related theories.

Classical		Quantum
compact Hausdorff space	$\longleftrightarrow$	unital C^*-algebra
compact group	$\longleftrightarrow$	compact quantum group
compact (spin) manifold	$\longleftrightarrow$	spectral triple
compact metric space	$\longleftrightarrow$	compact quantum metric space

- The theory is due to Rieffel and builds on that of *order unit spaces*.

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A PRIMER ON ORDER UNIT SPACES

- Initiated by Kadison in the 1950's.
- An *order unit space* is a unital (real) subspace V of the selfadjoint part of A_{sa} of a unital C^*-algebra A.
- The usual partial order on A_{sa} descends to one on V.
- There is an abstract definition as well, shown by Kadison to be equivalent to the one above.
- One may then define the *state space* of V:

$$\mathcal{S}(V) := \{\mu\colon V \to \mathbb{R} \mid \mu \text{ linear, bounded with } \|\mu\| = 1 = \mu(1)\}$$

- As in C^*-theory, states are automatically positive and $\mathcal{S}(V)$ is a weak*-compact subset of the dual space V'.

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DEFINITION (RIEFFEL, 1999)

An (order unit) compact quantum metric space is a pair (V, L) where V is an order unit space and $L\colon V \to [0,\infty)$ is a seminorm such that:

(i) *$L(a) = 0$ iff $a \in \mathbb{R}.1$*

(ii) *The quantity*

$$\rho_L(\mu,\nu) := \sup\{|\mu(a) - \nu(a)| : L(a) \leqslant 1\}, \qquad \mu,\nu \in \mathcal{S}(V)$$

defines a metric on $\mathcal{S}(V)$ which metrises the weak-topology.*

In this situation, L is called a Lip-norm.

- If (X,d) is a compact metric space then
  $$V = C_{\mathrm{Lip}}(X)_{\mathrm{sa}} := \{f \in C(X)_{\mathrm{sa}} : f \text{ is Lipschitz continuous}\}$$
  becomes a compact quantum metric space when endowed with $L(f) := \sup\left\{\frac{|f(x)-f(y)|}{d(x,y)} : x \neq y\right\}$.
- The metric ρ_L on $\mathcal{S}(V) = \mathrm{Prop}(X)$ is the so-called *Monge-Kantorovich metric* and $\rho_L\restriction_X = d$.

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C^*-ALGEBRAIC CQMS

- The setting of order unit spaces may seem strange at first, and one can also take a more C^*-algebraic approach:

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Let A be a unital C^*-algebra equipped with a seminorm $L\colon A \to [0, \infty]$ (allowed to be infinite) satisfying that $L(x^*) = L(x)$ for all $x \in A$. Then (A, L) is called a C^*-algebraic CQMS if

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- If one just assumes (i) and (ii) then it can be checked that (A, L) is a C^*-algebraic CQMS iff $(V_{sa}, L\restriction_{V_{sa}})$ is an order unit CQMS, and $\mathrm{res}\colon \mathcal{S}(A) \to \mathcal{S}(V_{sa})$ is a bijective isometry.

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Examples from NCG

- If $(A, \mathcal{H}, D)$ is a spectral triple, then sometimes – but not always – one obtains a CQMS by defining

$$L(a) := \|[D, a]\|$$

  for those $a \in A$ for which $[D, a]$ extends boundedly to $\mathcal{H}$. In this case one has a compact *spectral metric space* [Bellissard-Marcolli-Reihani].

- This is the case when $A = C(M)$ for a compact, connected, Riemannian spin manifold and D is the Dirac operator, in which case the metric ρ_L on $\mathcal{S}(C(M))$ restricts to the Riemannian metric on $M \subset \mathcal{S}(C(M))$ [Connes].

- When Γ is a word hyperbolic group (or $\mathbb{Z}^n$) equipped with a length function ℓ then $D_\ell(\delta_\gamma) := \ell(\gamma)\delta_\gamma$ turns $(C^*_{\text{red}}(\Gamma), \ell^2(\Gamma), D_\ell)$ into a spectral metric space [Ozawa-Rieffel].

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GROMOV-HAUSDORFF DISTANCE

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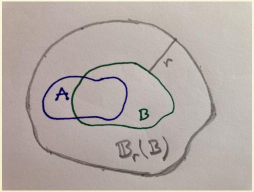
- Then their *Hausdorff distance* is defined by

$$\mathrm{dist}_{\mathrm{H}}^{d}(A, B) := \inf\{r > 0 \mid A \subset \mathbb{B}_r(B) \text{ and } B \subset \mathbb{B}_r(A)\}$$



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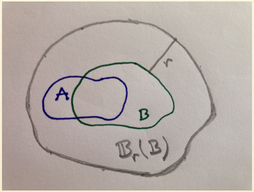

- And for two compact metric spaces (X_1, d_1) and (X_2, d_2) their *Gromov-Hausdorff distance* is defined as

$$\mathrm{dist}_{\mathrm{GH}}(X_1, X_2) := \inf_{d} \left\{ \mathrm{dist}_{\mathrm{H}}^{d}(X_1, X_2) \right\}$$

where the infimum runs over all metrics on $X_1 \sqcup X_2$ restricting to d_1 and d_2 respectively.

$\rightsquigarrow$ quantum-GH-distance

Quantum Gromov-Hausdorff distance

- If (V_1, L_1) and (V_2, L_2) are order unit CQMS then a seminorm $L\colon V_1 \oplus V_2 \to [0, \infty)$ is called *admissible* if L is a Lip-norm and the induced quotient seminorms on V_1 and V_2 agree with L_1 and L_2.
- The two projections $V_1 \oplus V_2 \to V_i$ dualise to injections $\mathcal{S}(V_i) \to \mathcal{S}(V_1 \oplus V_2)$.
- And Rieffel then defines

$$\mathrm{dist}^q_{\mathrm{GH}}(V_1, V_2) := \inf\left\{\mathrm{dist}^{\rho_L}_{\mathrm{H}}(\mathcal{S}(V_1), \mathcal{S}(V_2)) : L \text{ admissible}\right\}$$

- This is symmetric, satisfies the triangle-inequality, and distance zero is equivalent to Lip-isometric isomorphism (at the level of completions).
- The map $C_{\mathrm{Lip}} : (X, d) \mapsto (C_{\mathrm{Lip}}(X)_{\mathrm{sa}}, L_d)$ is a contraction, but *not* an isometry.
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- If (V_1, L_1) and (V_2, L_2) are order unit CQMS then a seminorm $L\colon V_1 \oplus V_2 \to [0, \infty)$ is called *admissible* if L is a Lip-norm and the induced quotient seminorms on V_1 and V_2 agree with L_1 and L_2.
- The two projections $V_1 \oplus V_2 \to V_i$ dualise to injections $\mathcal{S}(V_i) \to \mathcal{S}(V_1 \oplus V_2)$.
- And Rieffel then defines

$$\mathrm{dist}^{q}_{\mathrm{GH}}(V_1, V_2) := \inf\{\mathrm{dist}^{\rho_L}_{\mathrm{H}}(\mathcal{S}(V_1), \mathcal{S}(V_2)) : L \text{ admissible}\}$$

- This is symmetric, satisfies the triangle-inequality, and distance zero is equivalent to Lip-isometric isomorphism (at the level of completions).
- The map $C_{\mathrm{Lip}} : (X, d) \mapsto (C_{\mathrm{Lip}}(X)_{\mathrm{sa}}, L_d)$ is a contraction, but *not* an isometry.
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REMARKS ON THE DEFINITIONS

- For C^*-algebraic CQMS, one defines their quantum GH-distance as that of their (Lip-norm finite) self-adjoint parts.
- So distance zero only means that their order unit structures are identified in a Lip-norm preserving manner.
- More elaborate quantum-distances, due to Li, Kerr and Latrémolière, fix this problem, in that distance zero here implies isomorphism at the C^*-algebra level.
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Convergence and continuity results

- Non-commutative tori vary continuously [Rieffel].
- Fuzzy spheres (i.e. matrix algebras) converge to the classical 2-sphere [Rieffel].
- Approximation of non-commutative solenoids by non-commutative tori [Latrémolière-Packer].
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Crossed products

- If B is a (unital) C^*-algebra and $\beta\colon B \to B$ is an automorphism, then one can encode the dynamics of β in the crossed product $B \rtimes_\beta \mathbb{Z} := C^*(B, U)$.
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QUESTION A When is $B \rtimes_\beta \mathbb{Z}$ again a CQMS in a natural way?

QUESTION B How does $B \rtimes_\beta \mathbb{Z}$ vary in dist^q_{GH} if β varies?

- Question A was addressed by Bellissard-Marcolli-Reihani (2010) and Hawkins-Skalski-White-Zacharias (2013) in the setting of spectral metric spaces, with positive results if the action is *equicontinuous*: $\sup_{n\in\mathbb{Z}} L_B(\beta^n(b)) < \infty$.
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Theorem A (Kaad-K, 2019)

Let (B, V_B, L_B) be a C^*-algebraic QCMS such that L_B is lower semi continuous on its domain V_B, and assume that $\beta \in \mathrm{Aut}(B)$ satisfies that $\beta(V_B) = V_B$. Then for every $p \in \{1, \ldots, \infty\}$, the seminorm $L_p \colon c_c(\mathbb{Z}, V_B) \to [0, \infty[$ defined on $x := \sum_n x(n)U^n$ as

$$L_p(x) := \max\left\{ \left\| \sum_n nx(n)U^n \right\|, \|L_B \circ x\|_p, \|L_p \circ x^*\|_p \right\}$$

turns $B \rtimes_\beta \mathbb{Z}$ into a C^*-algebraic QCMS.

- In the case of spectral metric spaces, we also have more geometric criteria going beyond the equicontinuous case considered earlier, but with the price that the crossed products are, in general, only non-spectral CQMS.

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- In connection with Question B, we proved the following:

Theorem B

Let $(\beta_t)_{t \in T}$ be a family of automorphisms of B parametrised by a compact Hausdorff space T. Assume moreover that

- *L_B is lower semi continuous on V_B.*
- *$\beta_t(V_B) = V_B$ for all $t \in T$.*
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Then for each of the seminorms L_p from Theorem A, the family of $B \rtimes_{\beta_t} \mathbb{Z}$ of CQMS varies continuously in the parameter t with respect to the quantum Gromov-Hausdorff distance.

- As an example, Theorem B applies to a compact metric space (X, d) and a family $(\varphi_t)_{t \in T}$ in $\mathrm{Iso}(X)$ which is continuous for d_∞ on $C(X, X)$ by letting $\beta_t \in \mathrm{Aut}(C(X))$ be given by $\beta(f) := f \circ \varphi_t$.

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$\rightsquigarrow$ q-deformations

- In connection with Question B, we proved the following:

THEOREM B

Let $(\beta_t)_{t\in T}$ be a family of automorphisms of B parametrised by a compact Hausdorff space T. Assume moreover that

- *L_B is lower semi continuous on V_B.*
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- Perhaps surprisingly, some of the most central objects in NCG are not yet well understood from the QMS point of view — at least not in a way that reflects the geometry well.
- An example is Podleś' q-deformed 2-sphere S_q^2.
- Actually, only in 2018, Aguilar and Kaad showed that the Dąbrowski-Sitarz Dirac operator, D_q, turns S_q^2 into a spectral metric space.
- Recall that $C(S_q^2) := C(SU_q(2))^{\mathbb{T}}$
- Concretely, $C(S_q^2)$ is generated by a selfadjoint A_q and a non-selfadjoint B_q from $\mathrm{Pol}(SU_q(2))$.

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- One has $\sigma(A_q) := \{0\} \cup \{q^{2k} : k \in \mathbb{N}_0\}$ and $\sigma(A_1) = [0, 1]$, so $\sigma(A_q)$ can be thought of as quantised intervals.
- And $C(\sigma(A_q)) \simeq C^*(A_q) \subset C(S_q^2)$ inherits the structure of a compact quantum metric space, which therefore defines a metric d_q on $\sigma(A_q)$.
- Similarly, $C([0,1]) \simeq C^*(A_1) \subset C(S^2)$ inherits the structure of a compact quantum metric space, and hence defines a metric d_1 on $[0, 1]$.
- It is not difficult to check that $([0,1], d_1)$ is isometrically isomorphic to $[0, \pi]$ with the standard metric $d_{\mathbb{R}}$ from $\mathbb{R}$.
- In a recent joint work with Gotfredsen and Kaad we obtain a concrete formula for d_q, and show that

$$(X_q, d_q) \xrightarrow[q \to 1]{\mathrm{dist}_{\mathrm{GH}}} ([0, \pi], d_{\mathbb{R}})$$

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A quick glimpse into the future

- The previous result can be viewed as a 'proof of concept result' indicating that perhaps the quantised spheres S_q^2 converge to the classical sphere S^2 as q tends to 1.
- In a joint project with Jens Kaad and Konrad Aguilar, we are currently working on this problem.
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